Are diagrammatic proofs legitimate?

It is striking that although diagrammatic arguments were once respectable as parts of proofs ("the paradigm of rigor" [1, 345]), they have now fallen into disrepute. Most famously appearing in Euclid's *Elements*, diagrammatic arguments occur frequently not only as illustrations, but as parts of purported proofs. Of course, even today, few deny that diagrammatic arguments have no role whatsoever in mathematics. But according to the received view, diagrammatic arguments cannot be parts of proofs: diagrammatic proofs, which are only so-called, are illegitimate. My aim in this paper is to challenge the received view by suggesting an alternative but modest view: diagrammatic arguments can be (essential) parts of proofs: the case for the legitimacy of diagrammatic proofs is better than often supposed.

I define diagrammatic proofs as proofs that depend on diagrammatic arguments, and diagrammatic arguments as arguments that depend on one or more diagrams. Similarly, I define verbal proofs as proofs that depend on verbal arguments, and verbal arguments as arguments that depend on natural or formal language. Proofs are distinguished from other arguments in that proofs are necessarily sound arguments—valid arguments with sound premises.

The alternative view is modest for three reasons. First, I do not claim that all diagrammatic arguments are parts of proofs, since diagrammatic arguments can be valid or invalid like all other arguments. Second, I do not claim that diagrammatic arguments can by themselves constitute proofs. It suffices that diagrammatic arguments can be proper parts of proofs. Third, I do not claim that diagrammatic arguments are in general parts of proofs. It suffices that diagrammatic arguments can be parts of proofs for one or more claims, not that those claims must be proved with diagrammatic arguments.

In section 2, I discuss the nature of mathematical proof in general. Proof has two goals: to show *that* claims are true (or help us know they are true), and to show *why* they are true (to help us understand why they are true). I then invoke Hardy's [5] distinction between "official," formal proofs, which are rare, and "unofficial," informal proofs, which are most commonplace. I suggest that insofar as verbal proofs of the latter sort are legitimate in that they are *formalizable*, so too are diagrammatic proofs; the formalizability of diagrammatic proofs is established later.

In section 3, I outline the received view, which denies the legitimacy of diagrammatic proofs, and sees diagrammatic arguments only as heuristic devices and in need of translation to verbal arguments to earn legitimacy as proofs. This section discusses the views of Leibniz [7], Pasch [11], and Hilbert [6], and examples involving misleading arguments in number theory, geometry, and calculus. The main objections to the legitimacy of diagrammatic proofs are that they are (i) *unreliable*, (ii) *too particular*, and (iii) *not formalizable.*

In section 4, I suggest that with a more charitable approach, diagrammatic proofs may be almost as legitimate as more standard verbal proofs. I consider but find wanting Barwise and Etchemendy's [2] suggestion that diagrams in diagrammatic arguments somehow represent the claims they are about by exhibiting structural similarity—isomorphism or just homomorphism. I then more sympathetically adopt Brown's [3, 4] suggestion that although all proofs are visual in some respect, what matters is not their appearance or what they resemble, but what they represent—proofs are mere stepping stones that guide us to that "Aha!" moment where we see that a claim is true; a similar suggestion is found in Wittgenstein [13].

The crucial point I take away is that diagrams can not only represent particular claims, but general claims as well; the representations need not be pictorial, but can be symbolic, much as in verbal proofs. Further, poorly drawn diagrams in diagrammatic arguments are no more of an obstacle to proof than bad handwriting is in verbal arguments. These two points weaken the objections to the legitimacy of diagrammatic proofs on the basis that they are unreliable and too particular.

I then explore Mumma's [9, 10]'s formalization of Euclid's proofs in a formal system **Eu** with the help of Manders' [8] distinction between *exact* and *co-exact* properties of diagrams; while exact properties are relations between magnitudes (e.g., lengths) of the same type, coexact properties are topological relations between such magnitudes (e.g., points of intersection). It is suggested that diagrammatic arguments can be parts of proofs, provided that they depend only on the coexact properties of diagrams. This point weakens the objection to the legitimacy of diagrammatic proofs on the basis that they are not formalizable.

In section 5, I conclude that the answer to the titular question is not quite an outright *yes*, but a positive *maybe*. I then make some brief remarks about "free rides," a notion due to Shimojima [12] whereby diagrams allow us to make valid inferences not immediately apparent from certain assumptions (using the ordering of points on a line as an example), and areas of further research.

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