Are Forcing Axioms natural axioms?

The current situation in the foundation of set theory

After the invention of the method of forcing, by Paul Cohen in 1963 ([1]), set theory changed deeply and became what it is nowadays: a theory of independence proofs. Indeed this method allows to prove consistency results of the form Con(ZFC + $H + \varphi$) and Con(ZFC + $H' + \neg\varphi$), where Con(T) is a sentence asserting that the theory T is consistent, while H (respectively H') is a set theoretical hypothesis that acts like a sufficient condition for the proof of φ (respectively $\neg\varphi$), assuming ZFC as the background theory.

The huge amount of independence proofs obtained in the last decades undermined the convention that the axioms of ZFC could give a clear description of the universe of set theory. Consequently, these results gave rise to the search for new axioms able to eliminate the relativism given by the phenomenon of independence. The philosophical rationale behind this position is a strong form of semantical realism. In fact the advocates of this position assume the existence of a well-defined truth value for any mathematical statement. Consequently this position gave rise to what is normally called Gödel's program.

> these axioms [of infinity] show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far. (*Gödel. What is Cantor's continuum problem?*)

A general philosophical problem of this position is that it proposes to find axioms or general principles that should be accepted "without arbitrariness", that is in an objective way. In fact, in assuming a new axiom or a new principles the criteria of its justification can fall roughly into two different categories:

- *intrinsic reasons*: the justification is in some sense necessary because of the nature of the problem. The source of justification, in this case, can be the nature of the mathematical objects involved, or it may originate from the concept of set itself, shaped in the cumulative hierarchy form. There is a wide range of philosophical positions that can motivate the search for intrinsic reasons, but they all share a weak form of realism. Indeed a principle can be justified with intrinsic reasons once it mirrors an independent-existent fact. The latter can be the existence of a mathematical object, the absoluteness of some mathematical truth or even the nature of a concept. In short intrinsic reasons outline some necessary element of our mathematical work.

- *extrinsic reasons*: the burden of justification is left to the success of a set-theoretical principle. The philosophical attitude that motivates the proposal of these kinds of arguments is a form of quasi-empiricism applied to mathematics. The success of an axiom or of a principle shows its sufficient character is proving many different mathematical results.

Forcing Axioms

In my presentation I would like to present and justify the acceptance of the Forcing Axioms as new principles able to complete ZFC in a natural way. They are axioms that can be seen as topological principles that generalize Baire category theorem. From an intuitive point of view they assure the existence in V (the universal class of all sets defined by the formula $\varphi(x) = \{x : x = x\}$) of objects whose existence cannot be proved nor disproved in ZFC only. In particular, if we allow the possibility that there may be forcing extensions of the universal class V, by means of a notion of forcing with some particular properties, then the Forcing Axioms ensure that the new mathematical objects that we may find in such generic extensions should be already existent in V. In some sense Forcing Axioms saturate the universe of all sets by means of the possibility given by the method of forcing. Hence their name. As a consequence we could thing of a model of a Forcing Axiom as a model obtained after many forcing extensions. The general definition of a Forcing Axiom is the following.

Definition 0.1. *$FA(\Gamma, \kappa)$: For every notion of forcing $\mathbb{Q}$ with the property Γ and $\mathcal{D}$ a collection of κ-many dense subsets of $\mathbb{Q}$, there is a $\mathcal{D}$-generic filter G that intersects every $D \in \mathcal{D}$.*

I will briefly present the most used Forcing Axioms (i.e PFA and MM) and show their success in solving many problems independent from ZFC; among them the Continuum Hypothesis (CH). Moreover I will present a recent result by Viale (see [4]) that shows the necessary and sufficient character of a strengthening of MM in the solution of every problem that we can formulate in $H(\aleph_2)$: the structure of sets whose cardinality is hereditarily less or equal to $\aleph_1$. Once the extrinsic reasons for their acceptance will be ascertained I will argue in favor of their naturalness in order to show that we have also intrinsic reasons for their use. The starting point of this analysis will be the following interpretation of naturalness in mathematics (see [3] for a discussion of the philosophical problems related with this position).

> **Naturalness in mathematics**: a piece of mathematics is natural when it fits with a background idea that is relevant for the field under consideration.

Thus I will outline the philosophical idea behind Forcing Axioms: they ensure the existence of a mathematical object independently and prior to the existence of a model of set theory. Indeed the existential import of the Forcing Axioms is used to cook up new mathematical objects out of the generic filters, whose existence is guaranteed by these axioms, by means of a forcing argument. In short we have the following.

> **Main idea**: mathematical objects exist prior to their domain of definition.

Hence my task will be first to test if this idea is compatible with the contemporary notion of set exemplified by a cumulative hierarchy, in accordance to the work and conception of Zermelo. Secondly I will compare the Main idea with the notion of set that we find in the work of Cantor. In the first case I will find a substantial coherence that will allow me to say that Forcing Axioms are natural, while in the second case I will point to some conceptual difficulties at an epistemological level. In the end I will connect and discuss the Main idea with the so called quasi-combinatorialist attitude towards sets, as it is presented and analyzed in [2].

References

[1] J. P. Cohen. The independence of the continuum hypothesis. volume 50(6), pages 1143–1148, 1963.

[2] J. Ferreirós. On arbitrary sets and ZFC. *Bullettin of Symbolic logic*, 17(3):361–393, 2011.

[3] L. San Mauro and G. Venturi. Naturalness in mathematics: on the Static vs. Dynamic opposition. To appear in a volume of the *Boston Studies in the Philosophy of Science* (G. Lolli, M. Panza, G. Venturi eds.) Springer.

[4] M. Viale. Category forcings, MM^{+++}, and generic absoluteness for the theory of strong forcing axioms. To appear.