Category theory and set theory: algebraic set theory as an example of their interaction

I will begin by reviewing basic arguments and counterarguments about either (ZFC-style) set theory or category theory as a possible foundational theory for mathematics. I will then devote the main part of my talk to an important example of positive interaction between both theories, in connection with algebraic set theory.

Several arguments have been raised against category theory from a set-theoretic perspective: 1) The very notion of category presupposes the notion of set. 2) Most mathematical notions are defined in set-theoretic terms. 3) The Yoneda lemma itself proves that any category can be embedded into a category of sheaves of sets. 4) The objects and arrows that a category is made of are set-theoretic constructions: the category itself is but a superstructure built on top of those.

Symmetrically, several arguments have been raised againts set theory from a category-theoretic perspective: 1) If you fix a base topos S and work with toposes above S, the internal language of S becomes the standard set theory, so ZFC is but one particular case of set theory. 2) Set theory should not be confused with ZFC. There is a sense in which category theory gives rise to the genuine set theory[1]. 3) Working with categories allows one to get a better grasp of theories and to have a chance to account for modern mathematics.

In the first part of my talk, I will tackle those two sets of objections, and provide answers from both sides. But the hostility between set theory and category theory should be overcome. Championing either set theory or category theory is just extrapolating a tradition, or a certain field within mathematics (analysis and logical semantics in the case of set theory, algebraic geometry and modern algebraic topology in the case of category theory). "Foundations" are provided neither by set theory nor by category theory[2]. In this paper, I shall argue that a new form of collaboration can be foreseen and favored, thanks to Algebraic Set

[1]See Tom Leinster, "Rethinking set theory", arXiv:1212.6543.

[2]In a way, the opposition between set theory and category theory reactivates the old opposition between the syntactic, axiomatic approach to logic, with its basic notion of provability, and the semantical approach, with its basic notion of validity.

Theory (AST), and that this theory partly results from the *graft* of the theory of fibered categories onto ZFC.

Two different approaches toward AST have been developed up to now: a logical one (Alex Simpson, Steve Awodey), a geometric one (André Joyal, Ieke Moerdijk). I will focus on the original, geometric, one, and in particular of the first three axioms of Joyal-Moerdijk axiomatization. Let C be a Heyting pretopos C, Joyal-Moerdijk's idea is to characterize a class S of "small maps" in C, an arrow being a small map if all its fibers have a set-like size. Then, an object X of C is said to be *small* (set-like) if $X \to 1$ is a small arrow.

I will briefly set out all the axioms that define a class of small maps, and the fundamental notion of "ZF-algebra" that is drawn on their basis. The aim pursued by AST is to show that many models of ZF are of the form $V_{\langle C,S \rangle}$ for a certain class of small maps $\langle C, S \rangle$. I will then explain the very term of "descent" used by Joyal and Moerdijk (without any explanation) about their third axiom. Descent theory has been launched by Grothendieck in the context of "fibrations" (or "fibered categories"). A fibered category is the category-theoretic generalization of a surjective transformation. (I will give details.) Given a category C, let $C^{\to}$ be the category of all arrows in C. The codomain functor cod : $C^{\to} \to C$ then defines a fibration over C, the so-called "codomain fibration" attached to C. (The fiber above each $S \in$ Ob C is the category C/S composed of all arrows in C with codomain S.) That point establishes the connection of Joyal and Moerdijk's AST with Grothendieck's theory of fibrations: A class S of small maps is thought of as a *sub-fibration* $p_S : S \to C$ of the codomain fibration cod : $C^{\to} \to C$.

Now, descent theory[3] designates the abstract theory of glueing processes of local data in geometry and topology. It supplies the framework geared to describe the conditions that allow one to glue items lying above some category C, starting with a fibration whose base is C. A "descent morphism" in C is an arrow of C along which the glueing process can be performed. The first basic result of descent theory[4] is that if C has finite products and pullbacks, then the descent morphisms in C are exactly the universal strict epimorphisms of C. I will show that the first three axioms of AST can be fully understood as guaranteeing the same result in the context of the sub-fibration $p_S : S \to C$.

I will conclude that AST embodies a very nice combination of set theory and (fibered) category theory. From a set-theoretic viewpoint, any map is in fact a set: there is no arrow. The only arrows are the edges of the membership graph. The algebraic set-theoretic point of view, on the contrary, consists in putting to the fore arrows, along the "codomain fibration" – which is typical of the category-theoretic

[3]See A. Grothendieck, *Fondements de la géométrie algébrique*, exposé n°190, "Technique de descente et théorèmes d'existence en géométrie algébrique"(1959).

[4]See Grothendieck, art. cit., Proposition 2.1

point of view. The theory of fibrations thus reinterprets set theory so as to turn it into an arrow-based theory, rather than an object-based theory. That is why it can be described as a graft of category theory onto ZFC, which is much more fruitful than the standard rivalry between set theory and category theory.