Modality, abstract structures and second-order logic

David Lewis famously argued that from the truism that there were many ways the world could have been it followed that there were *ways*, and construed these ways as other possible worlds. If one is sympathetic to structuralism in the philosophy of physics one is more likely to take the ways to be structures the world could have embodied, and if one is also suspicious of quidditism, one will think in terms of *abstract* structures. But abstract structures seem to be the subject matter of mathematics, and thus mathematics gets its ontology from some rather simple reflections on modality.

This uses Quine's criterion of ontological commitment. Quine himself rejected modality and applied the criterion only once everything had been tucked into the procrustean bed of first-order logic. But once we do accept modality, it brings an ontology of possibilities with it. Lewis took the possibilities to be worlds; I take them to be structures. The advantage is that I can regard modal truths and truths about *abstracta* as equivalent. We can either say that the world *could have been* a bit different, or we can say that *there is* an abstract structure similar to that of the world, but a bit different; as neo-Fregeans might say, these are two ways of carving the same thought. And it follows that if there is a Benacerrafian epistemological problem about abstract structures then there is the same problem about modality, or, contraposing, an epistemology for modality (which most of us think must be possible) would provide one for mathematics.

In the world of abstract structures, unlike the world of concrete objects, *posse* is *esse*: every abstract structure that could exist does exist. For if no abstract structure could exist this is trivially true, and if abstract structures could exist it is absurd to think that some just happen to exist and others not, as though God might have created the projective plane but forgotten about the Klein 4-group. We are being Platonist rather than Aristotelian in that abstract structures do not need to be concretely exemplified to exist; indeed that is where we started from, with the structures the world could have exemplified but doesn't.

Logicians know what structures are: a structure is a domain of objects together with zero or more distinguished relations on the domain. In the case of completely abstract structures the only difference is that the objects are propertiless 'points'. These propertiless objects must be granted primitive identities so that they are not each other, so identity of indiscernibles fails. (Maybe it holds in any concrete instantiation of an abstract structure, but also maybe not, due to examples from quantum mechanics.)

Any finite structure can in principle be fully described and identified by giving its objects names and listing its relations. Skolem-Löwenheim tells us that we can never uniquely identify an infinite structure so long as we stay in first-order logic, but most of the important infinite structures of mathematics - $\mathbb{N}$ and $\mathbb{R}$ just for starters - can be uniquely identified via second-order axiomatizations. Of course almost all infinite structures are too chaotic to have any finite description.

Mathematics needs structures of higher order; already topology involves a predicate ('open') applied to subclasses of the domain. We could admit higher-order structures and higher-order logic to talk about them, but we don't need to, because we can use set theory, regarded as the theory of those domains with a single dyadic relation (called '$\in$') satisfying the axioms of ZF, and that is of course exactly what mathematicians do.

As is well-known, second-order ZF is quasi-categorical: its models vary only in height, but unfortunately we can't study them directly due to incomplete proof procedures. Set theorists thus study the much more varied models of first-order ZF (and the models they consider - e.g. inner models or models created by forcing - are almost never models for second-order ZF).

CH is true iff it is entailed by the axioms of second-order ZF, so if second-order entailment is determinate, CH has a determinate truth value. If you're silly enough to think that the semantics of second-order logic should be done in first-order set theory, then you'll conclude that it is very indeterminate (different models of first-order ZF give different accounts of second-order entailment). But on the view being put forward here the facts about second-order satisfiability/entailment are determined by *what abstract structures there are*. It is *not* that the structures all exist because they are to be found in some universe of set theory. Perhaps each one of them is quasi-concretely exemplified in some model of set theory, but its existence is already guaranteed by its abstract possibility (and it is abstract possibility that guarantees the existence of the model of set theory, an abstract structure with a distinguished binary relation).

I sidestep a full treatment of the determinacy of second-order logic (essentially the question of whether 'every possible subset' of an infinite set has determinate reference) and remark only that from the platonistic point of view adopted here out of the many models for first-order ZF we can identify those which are models of second-order ZF as (roughly) *the fattest*. That is enough for our purposes.

This kind of structuralist Platonism thus provides bivalence for almost any mathematical conjecture that can be expressed in the language of set theory. CH is either true in all models or false in all models. 'There is a measurable cardinal' is false in the shorter models, but if it is true in some model (i.e. if ZF^2+MC is satisfiable) then it stays true in all taller ones. But because we do not assume the existence of a supermodel (a Cantorian universe of absolutely all possible sets as opposed to all the determinately sized Zermelovian universes), a few sentences which switch on and off for ever will not have truth values.

Of course determinacy of truth value *in no way* entails the knowability of that truth value.