Multiversism and the Problem of Reference: How much Relativism is acceptable?

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Abstract

Since the 1930s the development of set theory has been marked by a significant phenomenon; independence proofs. Broadly speaking, there have been two reactions to this state of affairs; those that think that statements concerning 'width' (*id est* what subsets are formed at successor stages) are nonetheless bivalent, and those that do not. For this reason, views of this ilk provide excellent case studies for analysing the nature of truth, proof, and reference in mathematics.

In this paper I focus on one view that does not regard all questions of 'width' in set theory as bivalent. To see from where the view gains its motivation, it will be useful to first examine its polar opposite. We begin first with the following assumption to clarify the dialectic of the debate:

> [Ontological Descriptivism] Truth in set theory is simply a matter of ontology and how we refer to said ontology.

We see then that the view that *every* statement of set theory is bivalent can be reduced to the following claim about the subject matter of set theory:

> [Absolutism] There exists a unique, maximal, determinate universe of sets, to which we may refer precisely using our set theoretic concepts.

Absolutism thus ensures that every statement of set theory is determinately true or false, we simply have to look to the maximal universe of sets see that this is so. However, a problem arises when combined with the following principle:

> [Naturalism] A view of ontology on which one can transparently interpret a piece of set theoretic discourse where another view of ontology cannot is, in this respect, a better philosophy.

There then is a problem for Absolutism with respect to the independence results. It is not the mere fact independence (though this is somewhat of a challenge) that presents the biggest difficulty, but rather the manner in which the models for independence are constructed. In particular the use of *forcing* is problematic in this regard.

In order to force over a particular model of set theory $\mathfrak{M}$ we begin with a partial order $\langle \mathbb{P}, <_P \rangle \in \mathfrak{M}$. We then choose a filter G on $\mathbb{P}$ that intersects every dense subset D of P, such that $D \in \mathfrak{M}$. Next, we add G to $\mathfrak{M}$ and close under set-theoretic operations definable in $\mathfrak{M}$. The end result is a new model $\mathfrak{M}[G]$ that (given the correct choice of partial order and filter) satisfies the axioms and the negation of the sentence we wish to show to be unprovable. Importantly, however, $G \notin \mathfrak{M}$.

The challenge lies in the fact that often set theorists will perform forcing over V (the universe of *all* sets) itself. Since G cannot be in the model we are

forcing over, the Absolutist faces a problem. For, by Ontological Descriptivism, they should say that the statement “there is a forcing extension $V[G]$ such that...” is true in virtue of ontology (*id est* there is such a V-generic G). By Naturalism, they should be as transparent as possible with respect to forcing, avoiding paraphrase. However, while G is a set, G cannot be in V, but V is supposed to be all the sets there are. Something has to give.

The Multiversist solution to this is to deny that there is in fact a maximal universe of sets. They therefore assert the following:

> [Multiversism] There is no one universe of sets, but rather many. Any model of (some fragment of) a first-order set theory constitutes a universe of sets and is as legitimate as any other universe of sets.

Such a view has been proposed by Hamkins [Hamkins, 2012], and appears a very elegant philosophical theory, accounting for the practice of forcing wholly transparently.

I note, however, that a substantial question is left unanswered by the Multiversist; namely “how does reference to these universes occur?”. Hamkins suggests that each model corresponds to a concept of set. He speaks of fixing a set theoretic background, and then moving from universe to universe using model-theoretic constructions. However, Hamkinsian Multiversism is committed to a very strong form of *relativism* in that reference to these universes is inherently indeterminate. This is because our access to mathematical objects is given by first-order axiomatisations and these do not determine up to isomorphism a single model of set theory. Thus we should not really think of ourselves as fixing a *particular* set theoretic background; such reference is impossible. Rather we should think that we schematically refer to several universes, thereby identifying a ‘cloud’ of universes in the multiverse. It is this account of reference that creates a problem for the Multiversist in two distinct ways.

The first is that this account of reference opens the door to a referential regress. The model theoretic constructions possible are dependent upon the particular background of set in which one finds oneself, let it be denoted by ‘V_1’. Thus V_1 determines the extent of the multiverse. Given this, the exact cloud c_1 picked out by a given utterance of set theoretic statements is in fact dependent on V_1. But as it was acknowledged earlier, picking out an exact universe (V_1) is impossible. Thus it turns out that the initial selection of c_1 was dependent on a prior selection of a different cloud c_2. Again though, selection of c_2 is dependent on the extent of the multiverse, and hence on selection of a different universe V_2. But reference to V_2 is impossible, and hence this should really be analysed as reference to some cloud c_3. It is clear that there is no end to this process, and we have a non-well-founded dependency chain.

This would all be mitigated if the regress were non-vicious, however, unfortunately it is. The fact that the multiversist correlates set-concepts with models entails that, on a given occasion of set theoretic reference, we are in fact employing infinitely many concepts at once. This is epistemologically implausible.

The second problem is more technical in nature but appeals directly to reference. Let us concede for the moment that the Hamkinsian multiversist has a satisfactory way of picking out a particular initial starting set theoretic background $\mathfrak{M}$. The standard move when forcing is to define $\mathbb{P}$-names for ideal objects using the partial order then ascribe values to them using a $\mathfrak{M}$-generic G. As G is a set, it cannot be picked out uniquely, to admit so would be to admit that we have a determinate concept of set. Further, selection of an unintended generic will result in different statements being satisfied. However, from our current position in $\mathfrak{M}$ it is impossible to pick out any particular G, from the perspective of $\mathfrak{M}$ all $\mathfrak{M}$-generics are ideal objects. In this way forcing becomes highly intractable for the Hamkinsian multiversist.

I then use this failing of the Multiverse View to identify what is required for a philosophy of mathematics to have a satisfactory account of reference. I note that some views[1] do admit of some non-bivalence, and indeed some relativism. However, I argue that a core of definite concepts is required for a satisfactory account of the philosophy of mathematics. In particular, one must accept the definiteness of a sufficient number of concepts to facilitate reference to at least some models of one's theory. A relativism of the strength Hamkins accepts is problematic, but understanding why allows us to see what is needed for a satisfactory philosophy of mathematics.

References

[Hamkins, 2012] Hamkins, J. D. (2012). The set-theoretic multiverse. *The Review of Symbolic Logic*, 5(3):416–449.

[Zermelo, 1930] Zermelo, E. (1930). Uber grenzzahlen und mengenbereiche. *Fundamenta Mathematicae,*, N/A:N/A.

[1]For example the view (presented in [Zermelo, 1930]) that the subject matter of set theory is constituted by the natural models of second-order ZFC with the full semantics.