Abstract

There is an enduring question in the philosophy of mathematics that has, perhaps, received less attention than is appropriate. It is: *What is the significance of a categoricity theorem?* To be sure, many philosophers have addressed the question with respect to particular theories or as part of their argument for a philosophy of mathematics (most often one or another brand of mathematical structuralism). However, more general discussions are few and far between. The aim of this paper is to begin to fill some of the gaps in the debate.

One of the first general things that can be said about this question leads to an interesting observation. That is, in general, arguments for or against the significance of a categoricity theorem – with respect to a particular mathematical and/or philosophical theory – generally fall into two groups:

1. those for or against the claim that a categoricity theorem for a given mathematical theory T *vindicates a particular axiomatisation* of T (given some antecedent beliefs about T); and
2. those for or against the claim that a categoricity theorem for T *vindicates our antecedent beliefs* about T (given a particular axiomatisation of T).

This claim, of the dichotomy between kinds of argument surrounding the value of a categoricity theorem, leads us to observe an interesting interplay between realism, formal axiomatisations and model theory. As with all special issues in the philosophy of mathematics, one's stance on the significance of a categoricity theorem necessarily reflects one's level of commitment to, broadly, three types of realism: ontological, semantic, and conceptual (or "working"). Indeed, it would be difficult to see why anyone would even be interested in the philosophical significance, or lack thereof, of a categoricity theorem – a mere model-theoretic property – if they were not committed to at least one of either: (i) the objective existence of mathematical structures/objects; (ii) the determinate truth value of mathematical sentences; or (iii) the objective existence of mathematical concepts as used by the mathematician. Likewise, the issue will hold little interest for those not acquainted with, or interested in the choice between, various formal axiomatisations for any given mathematical theory. And since categoricity is a property pertaining to the models of a given formal axiomatisation, opinions on the significance of a categoricity theorem depend on an interest in model theory; for example, categoricity theorems for a second-order axiomatisation of a theory can be proven only with *full* semantics, and not with restricted, 'Henkin' semantics.

Thus, we can see that, regardless of which group one's argument about the significance of a categoricity theorem falls into – towards a vindication of either an axiomatisation, or some pre-theoretic beliefs – the necessary context is the interplay between the formal and the informal. We have axiomatisations and model theory on the one hand; and pre-theoretic beliefs and realist vs. anti-realist stances on the other.

Let us call the significance of a categoricity theorem with respect to vindicating an axiomatisation of a theory *axiomatic/formal* significance, and let significance with respect to vindicating some pre-theoretic beliefs about a theory be called *doxastic/informal* significance. Then we can identify, in general, six potential kinds of significance for a categoricity theorem for some mathematical theory T:

- **Axiomatic ontological significance**: Some axiomatisation Γ of T successfully picks out the objectively existing mathematical objects of T.
- **Axiomatic semantic significance**: Some axiomatisation Γ of T correctly assigns determinate truth values to the formal counterparts of T-propositions.
- **Axiomatic conceptual significance**: Some axiomatisation Γ of T successfully picks out the objectively existing T-concepts as used by mathematicians.
- **Doxastic ontological significance**: Some axiomatisation Γ of T confirms our antecedent belief in the objective existence of the mathematical objects of T.
- **Doxastic semantic significance**: Some axiomatisation Γ of T confirms our antecedent belief in the determinate truth values of T-propositions.
- **Doxastic conceptual signicance**: Some axiomatisation Γ of T confirms our antecedent belief in the objective existence of those T-concepts as used by mathematicians.

We discuss the above and argue that categoricity theorems are not necessary for axiomatic or doxastic semantic significance, and that they are not sufficient for axiomatic or doxastic ontological significance. We then examine where these results leave the proposed dichotomy between arguments surrounding axiomatic significance, and arguments surrounding doxastic significance, having seemingly collapsed it in the ontological and semantic cases. As it turns out, our categorisation may be upheld in the conceptual case, as we propose that a categoricity theorem is sufficient for axiomatic conceptual significance, but not so for doxastic conceptual significance. That is, given a commitment to conceptual realism about a theory T, we have reason to believe that a categoricity theorem for an axiomatisation Γ of T successfully picks out those concepts. For example, given conceptual realism about arithmetic, Dedekind's categoricity theorem for PA^2 allows us to pick out the concept of 'the natural numbers'. However, we suggest that given a commitment to some axiomatisation Γ of T, a categoricity theorem for Γ is not enough to affirm any pre-formal belief in the objective existence of T-concepts. Given PA^2, Dedekind's categoricity theorem is not sufficient for vindicating antecedent beliefs in the objective reality of arithmetical concepts.

In sum, it is shown that categoricity theorems are capable only of having limited significance. Although others have argued similarly, here we provide a novel and helpful generalised picture of (i) the kinds of argument that can be given for or against the significance of a categoricity theorem; (ii) the kinds of significance that can be attributed to or denied of a categoricity theorem, and how this relates to one's individual subscription to realism; and (iii) precisely where arguments promoting the significance of a categoricity theorem go wrong, given the present survey.