

What are Structural Properties?

Structural properties play a central role in modern philosophy of mathematics, in particular in the debate about mathematical structuralism, c.f. (Linnebo 2008) or (Shapiro 2008). Despite the obvious philosophical relevance of the notion, so far we are missing a formal explication of structural properties. Informally, structural properties are usually characterized in one of two ways: (i) as those properties expressible purely in terms of the primitive relations of a mathematical theory, or (ii) as those properties holding of all *structurally similar* mathematical objects. So far, neither approach has been made formally precise.

In this talk, we try to remedy this situation. We will present two formal explications of structural properties, corresponding to the two informal characterizations above. The first explication goes back to some remarks about structural properties by Rudolf Carnap (2000). In this approach, we identify the structural properties (of a given class of mathematical objects) with those properties that are *invariant* under isomorphic transformations. For example, according to this explication, a property is a structural property of graphs iff it is shared by all isomorphic graphs. Concrete examples are quick at hand. According to this explication, the property of having finitely many nodes, the property of being well-founded, or the property of being cyclic are all examples of structural properties of graphs. On the other hand, the property of having the natural numbers as nodes or the property of being Kurt Gödel's favorite graph, do not qualify as structural properties of graphs according to this explication. Thus this invariance explication gives us a notion of structural properties for a specific class of mathematical objects. Furthermore, the invariance explication can be extended in a natural way to account for structural properties of individual objects in an ambient mathematical structure: We can say that a property is a structural property of an object in an ambient structure iff the property remains invariant under automorphisms of the ambient structure. So, for example, according to this explication the property of not having a predecessor is a structural property of the number 0, while the property of being the set \emptyset is not.

The second explication we will discuss is based on the notion of *logical definability* in second-order languages. The idea is that a property of the objects of a specific mathematical theory (such as the theory of groups or the theory of graphs) is a structural property iff the extension of the property (i.e. the class of objects the property holds of) is second-order definable from the primitive vocabulary of the theory in question. In

particular, according to this approach a property P of a class of mathematical objects is structural iff there is a formula φ such that (i) φ is *informative* (that is, neither valid nor invalid) and (ii) φ only contains second-order logical vocabulary and vocabulary from the theory of the objects in question and (iii) the structures satisfying φ in a model theoretic sense are exactly the ones in the extension of P . So, according to this approach, the property of being an Archimedean field is a structural property of the real numbers, while being constructed via Dedekind-cuts is not.

By discussing these two explications, we wish to reach two goals: First, we wish to get clear on the metaphysical underpinnings of the notion of structural properties. The two approaches present two different pictures: According to the invariance-based approach, we will argue, structural properties are best understood in terms of *aboutness*. For this we will take David Lewis's (19988) theory of subject-matters and aboutness for sentences and apply it to properties. This will give us a natural notion of subject-matter and aboutness for properties. For example, according to this notion the property of being blue will be about the subject matter of having a certain color. In this setting, we will show that the structural properties according to the invariance approach are exactly the ones that are about the structure of mathematical objects. Thus, we argue, the invariance based approach is essentially a *semantic* approach to structural properties: what properties count as structural has to do with their subject matter. According to the definability-based approach, on the other hand, we will argue that structural properties are best understood in terms of truthmaking. For this we will show that the notion of a mathematical structure satisfying a formula exemplifies the axioms of truthmaker theory. By then passing from formulas to the extensions of properties of structures, we will obtain a natural notion of truthmakers for second-order definable properties. We will then show the structural properties according to the definability-approach are exactly the properties whose truthmakers are the structure of mathematical objects. Thus, we argue, the definability approach is essentially an *objectual* or *metaphysical* approach to structural properties: what properties count as structural is based in the nature of the objects exemplifying it.

The second aim of our talk is to understand the relation between the two explications of mathematical properties. As we will show, the two characterizations do *not* determine the same class of properties. Thus, they do not specify the same pre-theoretical notion of a structural property. More precisely, it can be shown that second-order definability implies invariance but not vice versa. We will discuss several examples of invariant but non-definable mathematical properties to illustrate this point.

From these observations, we will draw some philosophical conclusions. We will point out two possible responses to the situation presented above. The first one is to take second-order definability as *the* correct (or the most adequate) principle to demarcate structural from non-structural mathematical properties. An argument in this direction can be made based on informal philosophical counterexamples to the invariance-based approach, which we will briefly present. These counterexamples are instances of invariant properties in the above sense that should not classify as structural in any reasonable pre-

theoretical understanding of the term. More specifically, we present an argument that these properties should not count as structural since they do not hold of objects in virtue of their *internal* structural composition in the way definable properties do.

The second view is to adopt a *Carnapian* tolerant stance with respect to the choice of the definition of structural properties. In this view, neither one of the two explications provides us with the one correct account of structural properties. Nevertheless, both can be viewed to have their philosophical and mathematical merits. As will be shown, the plausibility of this tolerant position can be further strengthened by the fact that the two approaches tend to converge if definability is specified relative to richer and richer logical languages. In the limit case, that is definability in *infinitary* languages, the two explications can be shown to be equivalent.

References

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