Axiomatic definitions in the Peano School

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1 Short abstract

Based on a detailed analysis of the different types and uses of definitions made by Peano (by axioms, by abstraction, by operators, conditional, ...) and by some of the collaborators to the Formulario (Vailati, Pieri, Burali-Forti, Vacca, Padoa), it will be claimed that the adequacy of definitions is not a question that can be established independently of an analysis of the mathematical practice in which the definitions are used. The definition of a mathematical concept must be analyzed also in the light of the definitions of other concepts that might be derived from it, and the adequacy evaluated for all definitions altogether. Based on an original reading of Peano's axiomatics, the paper will provide examples to support the idea that the choice of the most appropriate definition is not only a logical-philosophical question (based on syntactic criteria or metatheoretical criteria) but also a mathematical choice related to the objectives the mathematician has while trying to solve a specific problem and to the consideration of an axiomatic theory as a whole.

2 Long abstract

The search for mathematical primitives, and more generally the discussion on the role of definitions in mathematics, was the most important objective of the Peano School. At the 1900 Paris International Conferences in Philosophy, Mathematics, and Psychology, the Italian group impressed Russell with clarity of language and reasoning. As a matter of fact, they presented no less than six papers on the topic of definitions. Peano introduced definitions as conventionally chosen equalities that determine the primitive concepts of a theory and simplify its language.[1] Burali-Forti discussed the difference between definitions by abstraction, definitions by postulates, and nominal definitions.[2] Padoa developed a definability criterion to verify whether a system of primitive symbols is irreducible, and presented two further papers on the principles of geometry, and on the definition of the field of natural numbers.[3] Vailati interpreted Brentano's tripartition of mental facts in representations, expectations and volitions as having a logical meaning, corresponding to the distinction between definitions, factual propositions and judgments of value.[4]

[1] Peano 1900.
[2] Burali-Forti 1901.
[3] Padoa 1901; Padoa 1902a; Padoa 1902b.
[4] Vailati 1901.

The paper will explain the complex distinction between implicit and explicit definitions that occur in these writings and that has engendered many confusions in the literature. There are two main notions of 'implicit definitions' in the Peano School. One refers to axioms used as ways to define the primitives of a theory. The other opposes implicit (or indirect) to explicit definitions. Explicit definitions are nominal definitions expressed in a canonical form, i.e. as an equality between definiens and definiendum that satisfies particular conditions. Non-explicit definitions include definitions by operators, by induction, by abstraction, and conditional definitions. Examples will be offered to show how different kinds of definitions, whose distinction is clearly presented by Burali-Forti in the 1894 edition of his *Logica matematica*, are actually used in various editions of the *Formulario* and possible explanations will be offered.

To keep track of alternative possible choices, Peano introduces from 1897 onwards the notion of possible definition (first abbreviated as [Df] , then as Df? and finally as Dfp), abandoning the idea of the unicity of concept ordering. The role of possible definitions is explained in the light of an understanding of Peano's axiomatics as a dynamic enterprise that does not aim neither to build a calculus nor to grant a secure foundation to a given mathematical theory, but as a means to increase conceptual analysis and rigor. It is claimed that definitions play a key role in Peano's project, and have not only a logical but also an epistemological value.

This claim is supported by the analysis of earlier mathematical writings by Peano, where the criticism of definitions is aimed at the search for a rigorous presentation of geometry and analysis. In *Definizione geometrica delle funzioni ellittiche*, Peano aims to provide a new definition of elliptic functions that is simpler and more similar to the usual definitions of circular functions.[5] In *Sulla definizione dell'area di una superficie*, the goal is to provide a definition of the arc of a curve and the area of a surface that allows one to understand the similarity between the two cases, despite the difference in size.[6] In *Sulla definizione del limite di una funzione* the goal is to provide a definition of limit such that the limit always exists.[7] In *Sur la définition de la dérivée* Peano compares two different definitions of the derivative of a function and observes that one definition is more general than the other, because it admits discontinuous derivatives, while according to the other definition, if a derivative exists, then it is necessarily continuous.[8]

The analysis of these writings shows not only that Peano's interest in definitions long predates 1900, but also that adequacy criteria for definitions include logical (syntactic and metatheoretic), epistemological and mathematical requirements. Of course, there are syntactic criteria, which require that all definitions be nominal definitions, i.e., equations satisfying certain criteria of homogeneity between the two members of the equality. But there are also metatheoretical criteria, as the requirement that primitive notions be irreducible, i.e., not definable from other concepts assumed to be primitive. Then, there are criteria that are based on some virtues of mathematical conceptual analysis and involve the right level of generality for a given definition or problem, or the possibility to preserve an analogy between definitions of one-dimensional and two-dimensional figures: this is particularly evident in earlier mathematical writings.

Finally, there are other criteria that are epistemological in nature: for example, the requirement that the definitions be as few as possible (Peano) or describe the invariant primitive concepts with respect to some transformations of a geometric theory (Pieri).

[5] Peano 1888, p. 255.
[6] Peano 1890.
[7] Peano 1892a.
[8] Peano 1892b.

Another epistemological requirement is related to Peano's axiomatic presentation of mathematical theories: a definition should not be evaluated as an isolated sentence but as a part of an axiomatic theory. Its adequacy also depends on the kinds of definitions that one must introduce afterwards for the derivative concepts. For example, if one introduces a definition by abstraction of rational numbers, one then finds oneself introducing a definition of the sum between rational numbers that does not meet the requirements of homogeneity. The choice of definitions is therefore not made only by considering the primitives and their irreducibility but must be the result of an overall evaluation of the mathematical formulas to which the proposed axiomatization leads. A similar explanation might be given for Peano's rejection of the definition of natural numbers as classes of classes, thereby showing Peano's non-philosophical reasons for rejecting the logicist definition.

The latter example shows the complexity of Peano's view on adequacy criteria, and makes sense of an often quoted passage by Peano, where he claims that the best definition is nothing else but the definition that each teacher prefers. The paper will claim that, although never declaring any specific interest in philosophy, Peano's mathematical practice suggests a precise view on the role definitions play in the development of mathematical knowledge. Peano shared with Vailati a pluralist, anti-dogmatic, and anti-foundationalist conception of definitions. Defending the plurality of views that emerged in his own school, Peano argued that definitions by abstraction, by a nominal definition, by means of operators, and conditional are equally rigorous: Vailati's and Peano's tolerance for different kinds of definitions does not coincide with the positions of Padoa, who favored a definition of rational number by abstraction, and Burali-Forti, who on the contrary criticized, at least at first, definitions by abstraction as based on intuitions rather than on concepts.

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